

QUALITY CONTROL FOR MARKOV CHAINS AND FREE BOUNDARY PROBLEMS ⁽¹⁾

BY

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ABSTRACT. A machine can manufacture any one of n Markov chains P_x^λ ($1 < j < n$); the P_x^λ are defined on the space of all sequences $x = \{x(m)\}$ ($1 < m < \infty$) and are absolutely continuous (in finite times) with respect to one another. It is assumed that chains P_x^λ evolve in a random way, dictated by a Markov chain $\theta(m)$ with n states, so that when $\theta(m) = j$ the machine is producing P_x^λ . One observes the σ -fields of $x(m)$ in order to determine when to inspect $\theta(m)$. With each product P_x^λ there is associated a cost c_j . One inspects θ at a sequence of times (each inspection entails a certain cost) and stops production when the state $\theta = n$ is reached. The problem is to find an optimal sequence of inspections. This problem is reduced, in this paper, to solving a certain free boundary problem. In case $n = 2$ the latter problem is solved.

0. Introduction. Let X be a fixed countable subset of the real line. Let $\theta(t)$ ($t = 0, 1, 2, \dots$) be a Markov chain with n states $1, 2, \dots, n$, and with transition probability matrix p_{ij} . With each state i we associate a Markov chain P_x^λ defined on the space Ω_1 of sequences $(x_0, x_1, x_2, \dots)$ where each x_i varies in X . We assume that the P_x^λ are distinct from each other and absolutely continuous (in finite time) with respect to one another. Denote by $E^{i,x}$ the expectation corresponding to the random evolution of the P_x^λ in accordance with the chain $\theta(t)$ starting at $\theta = i$ and x .

Let $K_1, \dots, K_{n-1}$ be given positive numbers. Let $c_1, \dots, c_n$ be given nonnegative numbers and define a function $f(\theta)$ by $f(i) = c_i$ if $i = 1, 2, \dots, n$. Let $\tau = (\tau_1, \tau_2, \dots)$ be an increasing sequence of "inspection times" in the sense that τ_i assumes only nonnegative integer values and each set $(\tau_i \leq s)$ (s nonnegative integer) depends only on the coordinates $x_0, x_1, \dots, x_s$ and on the knowledge of $\theta(\tau_j)$ for all $1 \leq j \leq i - 1$.

Throughout this paper we shall use the notation

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$$\int_a^b g(s) ds = g(a) + g(a+1) + \cdots + g(b) \quad (0.1)$$

where a, b are integers and $0 \leq a < b$.

Consider the cost function

$$J_x^i(\tau) = E^{i,x} \left[K_i + \sum_{j=1}^{n-1} K_j \left[\sum_{l=1}^{\infty} I_{\theta(\tau_l)=j} \right] \right] \\ + E^{i,x} \left[\int_0^{\tau_1-1} f(\theta(s)) ds + \sum_{j=1}^{n-1} \sum_{l=1}^{\infty} I_{\theta(\tau_l)=j} \int_{\tau_l-1}^{\tau_{l+1}-1} f(\theta(s)) ds \right]. \quad (0.2)$$

The problem considered in this paper is to find and characterize a sequence of inspection times $\bar{\tau} = (\bar{\tau}_1, \bar{\tau}_2, \dots)$ such that

$$J_x^i(\bar{\tau}) = \inf_{\tau} J_x^i(\tau). \quad (0.3)$$

This is called a *quality control problem*. The same problem in the case of continuous-parameter Markov processes was studied by the authors in [1], [2]. The problem was reduced to solving a certain elliptic quasi variational inequality (q.v.i.). We shall establish a similar reduction also in the present setting of Markov chains. Analogously to the q.v.i. of [1], [2] we shall obtain here a “discrete” q.v.i. In the special case where $n = 2$ we shall solve the q.v.i.

The development of this paper proceeds parallel to [2]. Some of the results follow similarly to [2], and these will be mentioned only briefly. There are, however, some novel features in the present Markov chain setting.

In §1 we introduce the random evolution process (x, θ) . We choose a model as in [2, Appendix] which displays very clearly the structure of this evolution.

In §2 we introduce the p -process and prove results analogous to Theorems 2.1, 2.2 of [2]. The quality control problem is introduced in §3, where it is reduced to solving a certain “discrete” q.v.i.

In §4 we solve the q.v.i. in case $n = 2$ under some monotonicity assumption.

1. The (x, θ) process. It will be convenient to denote the discrete parameter of various Markov chains by t ; thus the parameter t will take values $t = 0, 1, 2, \dots$. We fix a countable set X of points on the real line and denote by Ω_1 the space of all sequences $\omega = (x_0, x_1, x_2, \dots)$ with $x_i \in X$. Viewing ω as a function $x = x(t) = x(t, \omega)$ on the nonnegative integers with values in X , we write $x_t = x(t) = x(t, \omega)$, $t = 0, 1, 2, \dots$.

Let $\theta(t)$ be a Markov chain with n states $1, 2, \dots, n$ defined on a probability space Ω_0 of all sequences $\omega' = (\theta_0, \theta_1, \theta_2, \dots)$ where each θ_i may take values $1, 2, \dots, n$. Viewing ω' as a function $\theta = \theta(t) = \theta(t, \omega')$, we write $\theta_t = \theta(t) = \theta(t, \omega')$, $t = 0, 1, 2, \dots$. Denote the transition probability matrix of $\theta(t)$ by p_{ij} .

Let P_x^λ ($i = 1, \dots, n$) be n distinct Markov chains defined on Ω_1 and absolutely continuous (in finite time) with respect to one another. Denoting the transition probability matrix of P_x^λ by $p_{j,k}^\lambda$ we then have, for each pair (j, k) ,

$$\text{either } p_{j,k}^\lambda = 0 \text{ for all } 1 \leq i \leq n \text{ or } p_{j,k}^\lambda > 0 \text{ for all } 1 \leq i \leq n. \quad (1.1)$$

We are interested in an explicit construction of the random evolution of the P_x^λ in accordance with the law of $\theta(t)$. First we write down what, intuitively speaking, the transition probabilities should be:

$$\begin{aligned} P_{i,x}(\theta(t) = j, x(t) \in B) &= \sum_{\rho=0}^{t-1} \sum_{(i, \gamma_1, \dots, \gamma_\rho, j)}^* \\ &\cdot \sum_{u_i + u_{\gamma_1} + \dots + u_{\gamma_\rho} < t} p_{i,i}^{u_i-1} p_{i, \gamma_1}^{u_{\gamma_1}-1} p_{\gamma_1, \gamma_1}^{u_{\gamma_1}-1} p_{\gamma_1, \gamma_2}^{u_{\gamma_2}-1} \dots p_{\gamma_{\rho-1}, \gamma_\rho}^{u_{\gamma_\rho}-1} \dots p_{\gamma_\rho, j}^{t-u_i-u_{\gamma_1}-\dots-u_{\gamma_\rho}} \\ &\cdot P_x^\lambda \otimes_{u_i}^0 P_{x(u_i)}^\lambda \otimes_{u_i+u_{\gamma_1}}^{u_i} P_{x(u_i+u_{\gamma_1})}^\lambda \otimes \dots \\ &\quad \otimes_{u_i+u_{\gamma_1}+\dots+u_{\gamma_\rho}}^{u_i+u_{\gamma_1}+\dots+u_{\gamma_\rho}-1} P_{x(u_i+u_{\gamma_1}+\dots+u_{\gamma_\rho})}^\lambda (x(t) \in B) \end{aligned} \quad (1.2)$$

for $i = 1, \dots, n$; $x \in X$, where B is any subset of X . Here, the notation

$$\sum_{(i, \gamma_1, \dots, \gamma_\rho, j)}^* \quad \text{for } \rho \geq 1$$

means that summation is extended over all integers $\gamma_1, \dots, \gamma_\rho$ varying from 1 to n such that

$$i \neq \gamma_1 \neq \gamma_2 \neq \dots \neq \gamma_{\rho-1} \neq j;$$

for $\rho = 0$ it means that $i \neq j$, i.e., the sum is empty if $i = j$, and consists of one term if $i \neq j$. The summation

$$\sum_{\rho=0}^{t-1}$$

means that ρ varies over $0, 1, 2, \dots, t-1$ with one exception: if $i = j$ then there is no term with $\rho = 0$ and instead there appears the term

$$p_{i,i}^t P_x^\lambda (x(t) \in B);$$

we refer to this term as the term corresponding to $\rho = -1$. Finally, the notation

$$\sum_{u_i + u_{\gamma_1} + \dots + u_{\gamma_\rho} < t}$$

means that the summation is extended over all integers $u_i, u_{\gamma_1}, \dots, u_{\gamma_\rho}$ such that

$$u_i \geq 1, \quad u_{\gamma_1} \geq 1, \dots, u_{\gamma_\rho} \geq 1, \quad \text{and} \quad u_i + u_{\gamma_1} + \dots + u_{\gamma_\rho} \leq t.$$

The concept of the tensor product

$$P_x^1 \otimes_{u_1}^0 P_{x(u_1)}^2 \otimes \dots \otimes_{u_1+\dots+u_m}^{u_1+\dots+u_m-1} P_{x(u_1+\dots+u_m)}^{m+1}$$

used in (1.2) is the same as in [1], [2] (which is taken from [3]) with the obvious adaptation to the discrete parameter case.

Let $\Omega = \Omega_0 \otimes \Omega_1$ and denote by $\mathcal{F}_t$ and $\mathcal{M}_t$ the σ -fields generated by the first $t + 1$ coordinates of $(x_0, x_1, x_2, \dots)$ and of $\{(\theta_0, \theta_1, \theta_2, \dots), (x_0, x_1, x_2, \dots)\}$ respectively.

THEOREM 1.1. *The $P_{i,x}$ define a Markov process with respect to $\mathcal{M}_t$ and Ω .*

PROOF. It suffices to verify the Chapman-Kolmogorov equation

$$P_{i,x}(\theta(t) = j, x(t) \in B) = \sum_{l=1}^n \sum_y P_{i,x}(\theta(s) = l, x(s) = y) P_{l,y}(\theta(t-s) = j, x(t-s) \in B) \quad (1.3)$$

where s is any integer, $1 \leq s \leq t-1$, and B is any subset of X . The right-hand side of (1.3) is equal to

$$\begin{aligned} & \sum_{l=1}^n \sum_y \sum_{\mu=0}^{s-1} \sum'_{(i, \alpha_1, \dots, \alpha_\mu, l)} \sum'_{u_i + u_{\alpha_1} + \dots + u_{\alpha_\mu} \leq s} \sum_{\nu=0}^{t-s-1} \sum'_{(l, \beta_1, \dots, \beta_\nu, j)} \\ & \cdot \sum'_{u_l + u_{\beta_1} + \dots + u_{\beta_\nu} \leq t-s} P_{i,l}^{u_i-1} P_{l,\alpha_1}^{u_{\alpha_1}-1} \dots P_{\alpha_\mu,l}^{u_{\alpha_\mu}-1} \dots P_{l,l}^{s-u_i-\dots-u_{\alpha_\mu}} P_{l,l}^{u_l-1} P_{l,\beta_1}^{u_{\beta_1}-1} \\ & \cdot \dots P_{\beta_\nu,j} P_{j,j}^{t-s-u_l-\dots-u_{\beta_\nu}} P_x^{\lambda_x} \otimes_{u_i}^0 P_{x(u_i)}^{\lambda_{x(u_i)}} \otimes \dots \\ & \otimes_{u_i+\dots+u_{\alpha_\mu}-1}^{u_i+\dots+u_{\alpha_\mu}-1} P_{x(u_i+\dots+u_{\alpha_\mu})}^{\lambda_{x(u_i+\dots+u_{\alpha_\mu})}} (x(s) = y) \\ & \cdot P_y^{\lambda_y} \otimes_{u_l}^0 P_{x(u_l)}^{\lambda_{x(u_l)}} \otimes \dots \otimes_{u_1+\dots+u_{\beta_\nu}-1}^{u_1+\dots+u_{\beta_\nu}-1} P_{x(u_1+\dots+u_{\beta_\nu})}^{\lambda_{x(u_1+\dots+u_{\beta_\nu})}} (x(t-s) \in B). \quad (1.4) \end{aligned}$$

Summing over y and combining the two factors P^{λ_x} as in [2, following (A.7)] we deduce that the sum over y of the tensor products is equal to

$$\begin{aligned} & P_x^{\lambda_x} \otimes_{u_i}^0 P_{x(u_i)}^{\lambda_{x(u_i)}} \otimes \dots \otimes_{u_i+\dots+u_{\alpha_\mu}-1}^{u_i+\dots+u_{\alpha_\mu}-1} P_{x(u_i+\dots+u_{\alpha_\mu})}^{\lambda_{x(u_i+\dots+u_{\alpha_\mu})}} \otimes_{s+u_l}^{u_l+\dots+u_{\alpha_\mu}} P_{x(s+u_l)}^{\lambda_{x(s+u_l)}} \\ & \otimes \dots \otimes_{s+u_l+u_{\beta_1}+\dots+u_{\beta_\nu}-1}^{s+u_l+u_{\beta_1}+\dots+u_{\beta_\nu}-1} P_{x(s+u_l+u_{\beta_1}+\dots+u_{\beta_\nu})}^{\lambda_{x(s+u_l+u_{\beta_1}+\dots+u_{\beta_\nu})}} (x(t) \in B). \end{aligned}$$

Next we substitute $u_l + s \rightarrow u_l$. The sum

$$\sum'_{u_l + u_{\beta_1} + \dots + u_{\beta_\nu} \leq t-s}$$

becomes a sum

$$\sum'_{u_l + u_{\beta_1} + \dots + u_{\beta_\nu} \leq t, u_l \geq s+1}$$

where the prime “ ’ ” in the last summation indicates that $u_{\beta_1} \geq 1, \dots, u_{\beta_\nu} \geq 1$.

We next substitute $u_l \rightarrow u_l + u_{\alpha_1} + \dots + u_{\alpha_\mu} + u_l$. The last sum becomes a summation over $u_l, u_{\beta_1}, \dots, u_{\beta_\nu}$ subject to

$$\begin{aligned} & \sum'_{u_l + u_{\alpha_1} + \dots + u_{\alpha_\mu} + u_l + u_{\beta_1} + \dots + u_{\beta_\nu} \leq t} \\ & u_l \geq s+1 - u_l - u_{\alpha_1} - \dots - u_{\alpha_\mu} \end{aligned}$$

and the prime " ' " indicates that $u_{\beta_1} \geq 1, \dots, u_{\beta_r} \geq 1$.

The effect of the two substitutions is to transform (1.4) into the sum (cf. [2])

$$\begin{aligned} & \sum_{l=1}^n \sum_{\mu=0}^{s-1} \sum'_{(i, \alpha_1, \dots, \alpha_\mu, l)} \sum'_{u_i + u_{\alpha_1} + \dots + u_{\alpha_\mu} \leq s} \sum_{\nu=0}^{t-s-1} \sum'_{(l, \beta_1, \dots, \beta_r, j)} \\ & \cdot \sum_{\substack{u_i + u_{\alpha_1} + \dots + u_{\alpha_\mu} + u_l + u_{\beta_1} + \dots + u_{\beta_r} \leq t \\ u_l \geq s+1 - u_i - u_{\alpha_1} - \dots - u_{\alpha_\mu}}} p_{l,i}^{u_i-1} p_{i,\alpha_1} \dots p_{\alpha_\mu, l} p_{l,l}^{u_l-1} p_{l,\beta_1}^{u_{\beta_1}-1} p_{\beta_1, \beta_1} \dots \\ & p_{\beta_r, j} p_{j,j}^{t-u_i-u_{\alpha_1}-\dots-u_{\alpha_\mu}-u_l-u_{\beta_1}-\dots-u_{\beta_r}} p_x^{\lambda_x} \otimes_u^0 p_{x(u_i)}^{\lambda_{\gamma_1}} \otimes \dots \otimes_{u_i+\dots+u_{\alpha_\mu}-1}^{u_i+\dots+u_{\alpha_\mu}} \\ & \cdot p_{x(u_i+\dots+u_{\alpha_\mu})}^{\lambda_x} \otimes_{u_i+\dots+u_{\alpha_\mu}}^{u_i+\dots+u_{\alpha_\mu}} p_{x(u_i+\dots+u_{\alpha_\mu}+u_l)}^{\lambda_{\beta_1}} \otimes_{u_i+\dots+u_{\alpha_\mu}+u_l}^{u_i+\dots+u_{\alpha_\mu}+u_l} \\ & \cdot p_{x(u_i+\dots+u_l+u_{\beta_1})}^{\lambda_{\beta_2}} \otimes \dots \otimes_{u_i+\dots+u_{\beta_r}-1}^{u_i+\dots+u_{\beta_r}} p_{x(u_i+\dots+u_{\beta_r})}^{\lambda_x} (x(t) \in B). \end{aligned} \quad (1.5)$$

The left-hand side of (1.3) is equal to

$$\begin{aligned} & \sum_{\rho=0}^{t-1} \sum_{(i, \gamma_1, \dots, \gamma_\rho, j)} \sum'_{u_i + u_{\gamma_1} + \dots + u_{\gamma_\rho} \leq t} p_{i,i}^{u_i-1} p_{i,\gamma_1} p_{\gamma_1, \gamma_1}^{u_{\gamma_1}-1} p_{\gamma_1, \gamma_2} \dots \\ & p_{\gamma_\rho, j} p_{j,j}^{t-u_i-u_{\gamma_1}-\dots-u_{\gamma_\rho}} p_x^{\lambda_x} \otimes_u^0 p_{x(u_i)}^{\lambda_{\gamma_1}} \otimes \dots \\ & \otimes_{u_i+\dots+u_{\gamma_\rho}-1}^{u_i+\dots+u_{\gamma_\rho}} p_{x(u_i+\dots+u_{\gamma_\rho})}^{\lambda_x} (x(t) \in B). \end{aligned} \quad (1.6)$$

We have to prove that the expressions in (1.5) and (1.6) are equal.

Denote the general term under the summation in (1.6) by

$$I(i, \gamma_1, \dots, \gamma_\rho, j; u_i, u_{\gamma_1}, \dots, u_{\gamma_\rho}).$$

Then the general term under the summation in (1.5) is precisely

$$I(i, \alpha_1, \dots, \alpha_\mu, l, \beta_1, \dots, \beta_r, j; u_i, u_{\alpha_1}, \dots, u_{\alpha_\mu}, u_l, u_{\beta_1}, \dots, u_{\beta_r}).$$

Thus it remains to prove the following combinatorial lemma.

LEMMA 1.2. For any positive integers s, t with $s < t$,

$$\begin{aligned} & \sum_{l=1}^n \sum_{\mu=0}^{s-1} \sum'_{(i, \alpha_1, \dots, \alpha_\mu, l)} \sum'_{u_i + u_{\alpha_1} + \dots + u_{\alpha_\mu} \leq s} \sum_{\nu=0}^{t-s-1} \sum'_{(l, \beta_1, \dots, \beta_r, j)} \\ & \cdot \sum'_{\substack{u_i + u_{\alpha_1} + \dots + u_{\alpha_\mu} + u_l + u_{\beta_1} + \dots + u_{\beta_r} \leq t \\ u_l \geq s+1 - u_i - u_{\alpha_1} - \dots - u_{\alpha_\mu}}} I(i, \alpha_1, \dots, \alpha_\mu, l, \beta_1, \dots, \beta_r, j; \\ & u_i, u_{\alpha_1}, \dots, u_{\alpha_\mu}, u_l, u_{\beta_1}, \dots, u_{\beta_r}) \\ & = \sum_{\rho=0}^{t-1} \sum_{(i, \gamma_1, \dots, \gamma_\rho, j)} \sum'_{v_i + v_{\gamma_1} + \dots + v_{\gamma_\rho} \leq t} I(i, \gamma_1, \dots, \gamma_\rho, j; v_i, v_{\gamma_1}, \dots, v_{\gamma_\rho}). \end{aligned} \quad (1.7)$$

This lemma is entirely different from the corresponding combinatorial lemma used in [2].

PROOF OF LEMMA 1.2. Each term on the left-hand side of (1.7) corresponding to $\mu > 0$, $\nu > 0$ appears also on the right-hand side of (1.7) with

$$\gamma_k = \alpha_k \quad (1 \leq k \leq \mu), \quad \gamma_{\mu+1} = l, \quad \gamma_{\mu+m+1} = \beta_m \quad (1 \leq m \leq \nu),$$

$$v_i = u_i, \quad v_{\gamma_h} = u_{\gamma_h}. \quad (1.8)$$

The terms corresponding to $\mu = -1$, $\nu > 0$ arise when $l = i$, and then there are no α 's and

$$u_i + u_{\beta_1} + \cdots + u_{\beta_\nu} \leq t, \quad u_i \geq s + 1.$$

These terms also appear on the right-hand side of (1.7) (they are given by (1.8) with no α 's). Similarly, the terms with $\nu = -1$, $\mu > 0$ which appear on the left-hand side of (1.7) appear also on the right-hand side. Finally, the term corresponding to $\mu = -1$, $\nu = -1$ occurs only if $i = j$ and in that case it is precisely the term on the right-hand side of (1.7) corresponding to $\rho = -1$.

It remains to show that each term which appears on the right-hand side of (1.7) with $\rho \geq 0$ appears also on the left-hand side and that this correspondence is given by (the one-to-one mapping) (1.8).

Consider the case $\rho > 0$. Let

$$\sigma_0 = \inf\{\sigma; v_i + v_{\gamma_1} + \cdots + v_{\gamma_\sigma} \geq s\}.$$

Suppose first that

$$v_i + v_{\gamma_1} + \cdots + v_{\gamma_{\sigma_0}} = s. \quad (1.9)$$

If $\sigma_0 < \rho$ then define α 's, β 's and u 's by (1.8) with

$$l = \gamma_{\sigma_0+1}, \quad \mu = \sigma_0, \quad \nu = \rho - \sigma_0 - 1.$$

Since $v_i + v_{\gamma_1} + \cdots + v_{\gamma_{\sigma_0}} = s$ and $v_i \geq 1$, $v_{\gamma_1} \geq 1, \dots, v_{\gamma_{\sigma_0}} \geq 1$, we have $\mu \leq s - 1$. Similarly, since

$$u_{\gamma_{\sigma_0+1}} + \cdots + u_{\gamma_\rho} \leq t - (u_i + u_{\gamma_1} + \cdots + u_{\gamma_{\sigma_0}}) = t - s$$

and $\gamma_m \geq 1$, we must have $\nu \leq t - s - 1$. Therefore in order for the term $I(i, \gamma_1, \dots, \gamma_\rho, j; v_i, v_{\gamma_1}, \dots, v_{\gamma_\rho})$ to appear on the left-hand side of (1.7) we must show that the restriction

$$u_i \geq s + 1 - u_i - u_{\alpha_1} - \cdots - u_{\alpha_\mu}$$

is satisfied. But this follows immediately from (1.9) and the fact that $u_i \geq 1$.

If $\sigma_0 = \rho$ then the given term appears on the left-hand side of (1.7) with $l = j$, $\nu = -1$.

So far we have assumed that (1.9) holds. We now assume that (1.9) does not hold, i.e.,

$$v_i + v_{\gamma_1} + \cdots + v_{\gamma_{\sigma_0}} > s. \quad (1.10)$$

If $\sigma_0 > 0$ then we take $l = \gamma_{\sigma_0}$, $\mu = \sigma_0 - 1$, $\nu = \rho - \sigma_0$ in the definition (1.8). Since

$$u_i + u_{\gamma_1} + \cdots + u_{\gamma_{\sigma_0-1}} < s, \quad u_i \geq 1, u_{\gamma_h} \geq 1,$$

we have $\mu \leq s - 1$. Also

$$v_{\gamma_{s_0+1}} + \dots + v_{\gamma_p} \leq t - (v_i + v_{\gamma_1} + \dots + v_{\gamma_{s_0}}) < t - s$$

so that $\rho - \sigma_0 < t - s$, i.e., $\nu \leq t - s - 1$. Thus it remains to show that

$$u_i \geq s + 1 - u_i - u_{\alpha_1} - \dots - u_{\alpha_p}.$$

But this follows immediately from (1.10).

If $\sigma_0 = 0$ we take $l = i$ and proceed as in the last case. This completes the proof of the lemma.

Having proved Theorem 1.1, we denote by $P^{i,x}$ and $E^{i,x}$ the probabilities and expectations corresponding to the transition probabilities $P_{i,x}$. Recall that the probability space is Ω and that the σ -fields are the $\mathfrak{M}_t$.

We shall now extend formula (1.2).

LEMMA 1.3. *Let $A \in \mathfrak{F}_t$, $t = 0, 1, 2, \dots$. Then*

$$\begin{aligned} \int_A I_{\theta(t)=j} dP^{i,x} &= \sum_{\rho=0}^{t-1}^* \sum_{(i, \gamma_1, \dots, \gamma_{\rho}, j)} \sum_{u_i + u_{\gamma_1} + \dots + u_{\gamma_{\rho}} \leq t} \\ & p_{i,i}^{u_i-1} p_{i, \gamma_1}^{u_{\gamma_1}-1} p_{\gamma_1, \gamma_2}^{u_{\gamma_2}-1} \dots p_{\gamma_{\rho}, j}^{t-u_i-u_{\gamma_1}-\dots-u_{\gamma_{\rho}}} P_x^{\lambda} \otimes P_x^{\lambda}(u_j) \\ & \otimes_{u_i+u_{\gamma_1}}^{\lambda} P_x^{\lambda}(u_i+u_{\gamma_1}) \otimes \dots \otimes_{u_i+u_{\gamma_1}+\dots+u_{\gamma_{\rho}}}^{u_i+u_{\gamma_1}+\dots+u_{\gamma_{\rho}-1}} P_x^{\lambda}(u_i+u_{\gamma_1}+\dots+u_{\gamma_{\rho}})(A). \end{aligned} \quad (1.11)$$

PROOF. The proof is similar to the proof of Lemma A.3 in [2]. It suffices to prove (1.11) for a cylindrical set

$$A = (x(t_1) \in B_1, \dots, x(t_m) \in B_m), \quad t_1 < t_2 < \dots < t_m.$$

Let

$$A_i = (x(t_1) \in B_1, \dots, x(t_i) \in B_i), \quad 1 \leq i \leq m,$$

$$C_i = (x(t_{i+1} - t_i) \in B_{i+1}, \dots, x(t_m - t_i) \in B_m),$$

so that $A = A_m$. By the Markov property of the (x, θ) chain and by (1.2) we obtain, after substituting $u_i + t_{m-1} \rightarrow u_i$,

$$\begin{aligned} \int_A I_{\theta(t)=j} dP^{i,x} &= \sum_{l=1}^n p_{l,i}^{-t_{m-1}} \int_{A_{m-1}} I_{\theta(t_{m-1})=l} \sum_{\nu=0}^{t-1}^* \sum_{(i, \beta_1, \dots, \beta_{\nu}, l)} \\ & \cdot \sum_{\substack{u_i + u_{\beta_1} + \dots + u_{\beta_{\nu}} \leq t \\ u_i \geq t_{m-1} + 1}}^{\nu} p_{l,i}^{u_i-1} p_{l, \beta_1}^{u_{\beta_1}-1} \dots p_{\beta_{\nu}, j}^{t-u_i-u_{\beta_1}-\dots-u_{\beta_{\nu}}} P_x^{\lambda} \otimes P_x^{\lambda}(u_{t_{m-1}}) \\ & \otimes_{u_i-t_{m-1}}^0 P_x^{\lambda}(u_i-t_{m-1}) \otimes \dots \otimes_{u_i+\dots+u_{\beta_{\nu}-1}-t_{m-1}}^{u_i+\dots+u_{\beta_{\nu}}-t_{m-1}} \\ & \cdot P_x^{\lambda}(u_i+\dots+u_{\beta_{\nu}}-t_{m-1})(C_{m-1}) dP^{i,x}. \end{aligned} \quad (1.12)$$

Using the Markov property we can write the right-hand side in the form

$$\begin{aligned}
 & \sum_{k=1}^n \int_{A_{m-2}} I_{\theta(t_{m-2})=k} \sum_{l=1}^n p_{l,l}^{-t_{m-1}} \sum_{\mu=0}^{t_{m-1}-t_{m-2}-1} \sum'_{(k, \alpha_1, \dots, \alpha_\mu, l)} \sum_{\nu=0}^{t-t_{m-1}-1} \\
 & \cdot \sum'_{(l, \beta_1, \dots, \beta_\nu, j)} u_k + u_{\alpha_1} + \dots + u_{\alpha_\mu} < t_{m-1} - t_{m-2} \quad u_l + u_{\beta_1} + \dots + u_{\beta_\nu} < t \\
 & \cdot p_{l,l}^{t_{m-1}-t_{m-2}-u_k-\dots-u_{\alpha_\mu}} p_{l,l}^{u_l-t_{m-1}-1} p_{l,\beta_1} \cdot \dots \cdot p_{j,j}^{t_{m-1}-u_l-\dots-u_{\beta_\nu}} \\
 & \cdot P_{x(t_{m-2})}^{\lambda_k} \otimes_{u_k}^0 P_{x(u_k)}^{\lambda_{\alpha_1}} \otimes \dots \otimes_{u_k+\dots+u_{\alpha_\mu}}^{u_k+\dots+u_{\alpha_\mu-1}} \\
 & \cdot P_{x(u_k+\dots+u_{\alpha_\mu})}^{\lambda_k} \left[(x(t_{m-1} - t_{m-2}) \in B_{m-1}) \right. \\
 & \quad \cdot P_{x(t_{m-1}-t_{m-2})}^{\lambda_k} \otimes_{u_l-t_{m-1}}^0 P_{x(u_l-t_{m-1})}^{\lambda_{\alpha_1}} \otimes \dots \\
 & \quad \left. \otimes_{u_l+\dots+u_{\beta_\nu}-1}^{u_l+\dots+u_{\beta_\nu}-t_{m-1}} P_{x(u_l+\dots+u_{\beta_\nu}-t_{m-1})}^{\lambda_k} (C_{m-1}) \right] dP^{i,x}.
 \end{aligned}$$

We now make the substitution $u_k + t_{m-2} \rightarrow u_k$ which transforms

$$\sum'_{u_k + u_{\alpha_1} + \dots + u_{\alpha_\mu} < t_{m-1} - t_{m-2}} \quad \text{into} \quad \sum'_{\substack{u_k + u_{\alpha_1} + \dots + u_{\alpha_\mu} < t_{m-1} \\ u_k > t_{m-2} + 1}}$$

and then make the substitution $u_l - u_k - u_{\alpha_1} - \dots - u_{\alpha_\mu} \rightarrow u_l$ which transforms

$$\sum'_{u_l + u_{\beta_1} + \dots + u_{\beta_\nu} < t} \quad \text{into} \quad \sum'_{\substack{u_k + u_{\alpha_1} + \dots + u_{\alpha_\mu} + u_l + u_{\beta_1} + \dots + u_{\beta_\nu} < t \\ u_l > t_{m-1} + 1 - u_k - u_{\alpha_1} - \dots - u_{\alpha_\mu}}}$$

Using the rules (A.16), (A.17) of [2] we finally obtain

$$\begin{aligned}
 \int_A I_{\theta(t)=j} dP^{i,x} &= \sum_{k=1}^n p_{k,k}^{-t_{m-2}} \int_{A_{m-2}} I_{\theta(t_{m-2})=k} \sum_{l=1}^n \sum_{\mu=0}^{t_{m-1}-t_{m-2}-1} \\
 & \cdot \sum'_{(k, \alpha_1, \dots, \alpha_\mu, l)} \sum_{\nu=0}^{t-t_{m-1}-1} \sum'_{(l, \beta_1, \dots, \beta_\nu, j)} \sum_{\substack{u_k + u_{\alpha_1} + \dots + u_{\alpha_\mu} < t_{m-1} \\ u_k > t_{m-2} + 1}} \\
 & \cdot \sum'_{\substack{u_k + u_{\alpha_1} + \dots + u_{\alpha_\mu} + u_l + u_{\beta_1} + \dots + u_{\beta_\nu} < t \\ u_l > t_{m-1} + 1 - u_k - u_{\alpha_1} - \dots - u_{\alpha_\mu}}} \\
 & \cdot [p_{k,k}^{u_k-1} p_{k,\alpha_1} \cdot \dots \cdot p_{l,l}^{u_l-1} p_{l,\beta_1} \cdot \dots \cdot p_{j,j}^{t_{m-1}-u_l-u_{\alpha_1}-\dots-u_{\beta_\nu}} \\
 & \cdot P_{x(t_{m-2})}^{\lambda_k} \otimes_{u_k-t_{m-2}}^0 P_{x(u_k-t_{m-2})}^{\lambda_{\alpha_1}} \otimes \dots \\
 & \otimes_{u_k+\dots+u_{\alpha_\mu}-1}^{u_k+\dots+u_{\alpha_\mu}-t_{m-2}} P_{x(u_k+\dots+u_{\alpha_\mu}-t_{m-2})}^{\lambda_k} \otimes \dots \\
 & \otimes_{u_k+u_{\alpha_1}+\dots+u_{\beta_\nu}-1}^{u_k+u_{\alpha_1}+\dots+u_{\beta_\nu}-t_{m-2}} P_{x(u_k+u_{\alpha_1}+\dots+u_{\beta_\nu}-t_{m-2})}^{\lambda_k} (C_{m-2})] dP^{i,x}. \quad (1.13)
 \end{aligned}$$

We now apply a slightly different version of Lemma 1.2 whereby instead of $t_{m-2} = 0$ we have $t_{m-2} \geq 0$. We conclude that

$$\int_A I_{\theta(t)=j} dP^{i,x} = \sum_{k=1}^n p_{k,k}^{-t_{m-2}} \int_{A_{m-2}} I_{\theta(t_{m-2})=k} \sum_{\rho=0}^{t-t_{m-2}-1} \sum'_{(k, \gamma_1, \dots, \gamma_{\rho j})} \dots \sum'_{\substack{u_k + \dots + u_{\gamma_{\rho}} \leq t \\ u_k \geq t_{m-2} + 1}} [\dots] dP^{i,x} \quad (1.14)$$

where the expression in $[\dots]$ is the same as on the right-hand side of (1.13). Formula (1.14) is analogous to (1.12), except that $m-1$ has been replaced by $m-2$. Proceeding in this way step by step and setting $t_0 = 0$, $B_0 = X$, $A_0(x(t_0) \in B_0)$, $C_0 = A$, we arrive at (1.11) with $m-1$ and t_{m-1} replaced by 0 and t_0 respectively. But this relation is precisely the assertion of the lemma.

2. The p -process. In view of the assumption (1.1) we have

$$\frac{dP_x^{\lambda}}{dP_x^{\lambda}} \Big|_{\mathcal{F}_t} = \frac{P_{x,x(1)}^{\lambda} P_{x(1),x(2)}^{\lambda} \dots P_{x(t-1),x(t)}^{\lambda}}{P_{x,x(1)}^{\lambda} P_{x(1),x(2)}^{\lambda} \dots P_{x(t-1),x(t)}^{\lambda}} \quad (2.1)$$

on all paths for which both the numerator and the denominator do not vanish, and $P_x^{\lambda} = 0$, $P_x^{\lambda} = 0$ on all the remaining paths. Let

$$z_{ij}(s, t) = \frac{dP_x^{\lambda} \otimes_s^0 P_{x(s)}^{\lambda}}{dP_x^{\lambda}} \Big|_{\mathcal{F}_t} \quad (s < t). \quad (2.2)$$

Then we have

$$z_{ij}(s, t) = \frac{P_{x(s),x(s+1)}^{\lambda} P_{x(s+1),x(s+2)}^{\lambda} \dots P_{x(t-1),x(t)}^{\lambda}}{P_{x(s),x(s+1)}^{\lambda} P_{x(s+1),x(s+2)}^{\lambda} \dots P_{x(t-1),x(t)}^{\lambda}} \quad (s < t) \quad (2.3)$$

on the paths for which the numerator and denominator do not vanish. Clearly $z_{ij}(t, t) = 1$.

As in [2] we define

$$\bar{p}_{ij}(t) = P^{i,x}[\theta(t) = j | \mathcal{F}_t] \frac{dP^{i,x}}{dP_x^{\lambda}} \Big|_{\mathcal{F}_t}. \quad (2.4)$$

We then have (cf. [1], [2])

$$\begin{aligned} \bar{p}_{ij}(t) &= \sum_{\rho=0}^{t-1} \sum'_{(i, \gamma_1, \dots, \gamma_{\rho j})} \sum'_{u_i + u_{\gamma_1} + \dots + u_{\gamma_{\rho}} \leq t} p_{i,j}^{u_i-1} p_{i, \gamma_1}^{u_{\gamma_1}-1} p_{\gamma_1, \gamma_2}^{u_{\gamma_1}-1} \dots \\ &\dots p_{\gamma_{\rho j}, j}^{t-u_i-u_{\gamma_1}-\dots-u_{\gamma_{\rho}}-1} z_{i, \gamma_1}(u_i, u_i + u_{\gamma_1}) z_{i, \gamma_2}(u_i + u_{\gamma_1}, u_i + u_{\gamma_1} + u_{\gamma_2}) \\ &\dots z_{ij}(u_i + u_{\gamma_1} + \dots + u_{\gamma_{\rho}}, t). \end{aligned} \quad (2.5)$$

We now introduce the probabilities

$$\bar{p}^{p,x} = \sum_{i=1}^n p_i P^{i,x} \quad \left(p = (p_1, \dots, p_n), p_i \geq 0, \sum_{i=1}^n p_i = 1 \right) \quad (2.6)$$

and the process

$$X(t) = (x_x(t), p_1(p, t), \dots, p_n(p, t)) \quad (2.7)$$

where $t = 0, 1, 2, \dots$; $x_x(t)$ is $x(t)$ with $x(0) = x$ and

$$p_j(p, t) = \bar{E}^{p,x}[\theta(t) = j | \mathfrak{F}_t]. \quad (2.8)$$

Here $\bar{E}^{p,x}$ is the expectation corresponding to the probability $\bar{P}^{p,x}$. As in [2] we have

$$p_j(p, t) = \sum_{i=1}^n p_i \bar{p}_{i,j}(t) / \left[\sum_{l=1}^n p_l z_{i,l}(0, t) \sum_{k=1}^n \bar{p}_{l,k}(t) \right]. \quad (2.9)$$

THEOREM 2.1. *The process $X(t)$ is a Markov process, with respect to the σ -fields $\mathfrak{F}_t$ and the measures $\bar{P}^{p,x}$.*

The proof is similar to the proof of the corresponding result in the Appendix of [2] except that now we use Lemma 1.3 instead of Lemma A.2 of [2].

3. The quality control problem. Using the notation (0.1), we introduce the cost function (0.2) and, more generally, the cost

$$\begin{aligned} J_x^p(\tau) = & \bar{E}^{p,x} \left[K(p) + \sum_{l=1}^{\infty} K(\theta(\tau_l)) I_{\theta(\tau_l) \neq n} \right] \\ & + \bar{E}^{p,x} \left[\int_0^{\tau_1-1} f(\theta(s)) ds + \sum_{l=1}^{\infty} I_{\theta(\tau_l) \neq n} \int_{\tau_l-1}^{\tau_{l+1}-1} f(\theta(s)) ds \right] \end{aligned} \quad (3.1)$$

where $K(p) = K_i$ if $p = (p_1, \dots, p_n)$, $p_1 = \dots = p_{i-1} = 0$, $p_i \neq 0$; if the process $\theta(t)$ is such that $p_{i,j}(t) = 0$ whenever $j < i$ then no restrictions are made on the K_i , but if the process $\theta(t)$ can go in both directions then we require that $K_1 = K_2 = \dots = K_{n-1}$. Here $\tau = (\tau_1, \tau_2, \dots)$ is a sequence of inspection times, i.e.,

$$\tau_1 = \sigma_1, \quad \tau_{m+1} = \tau_m + \sum_{l=1}^{n-1} I_{\theta(\tau_m)=l} \sigma_{m,l}(\phi_{\tau_m}) \quad (m \geq 1) \quad (3.2)$$

where $\sigma_1, \sigma_{m,l}$ are stopping times with respect to $\mathfrak{F}_t$ with nonnegative integer values, and ϕ is the shift operator: $\phi_s x(t) = x(t+s)$. It is understood that $\tau_{m+i} = \infty$ ($i \geq 1$) on the set $\tau_m = \infty$. Also, in (3.1), $K(\theta(\tau_l)) I_{\theta(\tau_l) \neq n}$ and $\int_{\tau_l-1}^{\tau_{l+1}-1} f(\theta(s)) ds$ do not appear whenever $\tau_l = \infty$. We shall denote by $\mathcal{Q}$ the class of all sequences of inspection times. We are interested in the problem of characterizing $\bar{\tau}_p \in \mathcal{Q}$ such that

$$J_x^p(\bar{\tau}_p) = \inf_{\tau \in \mathcal{Q}} J_x^p(\tau). \quad (3.3)$$

Denote by $A_{x,p}$ the generator of the Markov process occurring in Theorem

2.1. Thus, $A_{x,p}$ is defined by

$$A_{x,p}u(x, p) = \bar{E}^{p,x}[u(x(1), p(p, 1)) - u(x, p)] \quad (3.4)$$

where $p(p, t) = (p_1(p, t), \dots, p_n(p, t))$.

Using the Markov property one can establish, by induction on t ($t = 1, 2, \dots$), Dynkin's formula

$$\begin{aligned} & \bar{E}^{p,x}[u(x_x(t), p(p, t))] - u(x, p) \\ &= \bar{E}^{p,x}\left[\int_0^{t-1} A_{x_x(s), p(p, s)}u(x_x(s), p(p, s)) ds\right]. \end{aligned} \quad (3.5)$$

We can now proceed as in [2] to reduce the problem of characterizing an optimal $\bar{\tau}_p$ as in (3.3) to the problem of solving the following quasi variational inequality (q.v.i.) for a function $V(x, p)$:

$$V(x, p) \leq K(p) + \sum_{j=1}^{n-1} p_j V(x, e_j) \quad (3.6)$$

where e_j is the j th unit vector $(0, 0, \dots, 0, 1, 0, \dots, 0)$,

$$A_{x,p}V(x, p) + \sum_{j=1}^n c_j p_j \geq 0, \quad (3.7)$$

$$\left[A_{x,p}V(x, p) + \sum_{j=1}^n c_j p_j \right] \left[K(p) + \sum_{j=1}^{n-1} p_j V(x, e_j) - V(x, p) \right] = 0 \quad (3.8)$$

where the p_j vary in the set $p_j \geq 0, \sum_{j=1}^n p_j = 1$ and x varies in X .

Let

$$\begin{aligned} S = \left\{ (x, p); x \in X, p = (p_1, \dots, p_n), p_j \geq 0, \sum_{j=1}^n p_j = 1, \right. \\ \left. V(x, p) = K(p) + \sum_{j=1}^{n-1} p_j V(x, e_j) \right\}. \end{aligned} \quad (3.9)$$

Define the $\mathcal{F}_t$ stopping times:

$$\sigma_*^p = \text{hitting time of the set } S \text{ by } X(t) = (x(t), p(p, t)),$$

$$\sigma_*^l = \sigma_*^p \text{ when } p = e_l,$$

$$\bar{\tau}_1^p = \sigma_*^p, \bar{\tau}_{m+1}^p = \bar{\tau}_m^p + \sum_{l=1}^{n-1} I_{\theta(\bar{\tau}_m^p)=l} \sigma_*^l(\phi_{\bar{\tau}_m^p}),$$

$$\bar{\tau}^p = (\bar{\tau}_1^p, \bar{\tau}_2^p, \bar{\tau}_3^p, \dots). \quad (3.10)$$

THEOREM 3.1. Let $V(x, p)$ be a solution of the q.v.i. (3.6) – (3.8). If the σ_*^p

are finite valued then

$$V(x, p) = \inf_{\tau \in \mathcal{Q}} J_x^p(\tau) = J_x^p(\bar{\tau}^p). \quad (3.11)$$

The proof is similar to the proof of the corresponding result in [2] and will therefore be omitted.

In the special case where

$$p_{ij} = 0 \quad \text{if } 1 \leq j < i \leq n \quad (3.12)$$

the q.v.i. reduces to a sequence of simpler q.v.i. analogous to (4.35)–(4.37) in [2].

Another type of simplification of (3.6)–(3.8) occurs when

$$p_{x(s), x(t)}^{\lambda} = p_{x(0), x(t-s)}^{\lambda} \equiv p_{t-s}^{\lambda} \quad \text{if } 0 \leq s < t. \quad (3.13)$$

In this case the numbers

$$\bar{P}^{p, x}[p_j(p, t) \in B; 1 \leq j \leq n]$$

do not depend on x and, consequently, the process

$$p_j(p, t) \quad (1 \leq j \leq n) \text{ with measures } \bar{P}^{p, 0} \quad (3.14)$$

is a Markov process. We shall denote its generator by A_p .

Denote by R_i^* the (countable) range of the process $(p_j(e_i, t); 1 \leq j \leq n)$ and let $R^* = \bigcup_{i=1}^{n-1} R_i^*$.

One is interested in the quality control problem mainly for the initial values $p = e_i$. In case (3.13) holds it then suffices to solve the q.v.i. in the set R^* only. Thus we have to solve a "discrete" q.v.i.

In the next section we shall solve the discrete q.v.i. in a case when $n = 2$.

4. Solution of the discrete q.v.i. in case $n = 2$. We assume that (3.12), (3.13) hold and that $n = 2$. Thus $p_{2,1} = 0$, $p_{2,2} = 1$. To rule out a trivial case, we assume that $p_{1,1} > 0$, $p_{1,2} > 0$.

Let

$$p_j^{\lambda_1} = p_{i,i+j}^{\lambda_1}, \quad p_j^{\lambda_2} = p_{i,i+j}^{\lambda_2} \quad (4.1)$$

where $p_{k,l}^{\lambda}$ is the transition probability matrix of P_x^{λ} . Denote by N the set of j 's for which $p_j^{\lambda} \neq 0$, and let

$$\mu_j = p_j^{\lambda_2} / p_j^{\lambda_1} \quad (j \in N). \quad (4.2)$$

Since

$$\sum_{j \in N} p_j^{\lambda} = 1 \quad (i = 1, 2), \quad (4.3)$$

we must have

$$\sum_{j \in N} p_j^{\lambda_1} \mu_j = 1. \quad (4.4)$$

Since $p_{2,1} = 0$, $p_{2,2} = 1$, we have

$$\bar{p}_{1,1}(1) = p_{1,1}, \quad \bar{p}_{1,2}(1) = p_{1,2}, \quad \bar{p}_{2,1}(t) = 0, \quad \bar{p}_{2,2}(t) = 1,$$

Hence, (2.9) for $t = 1$ simplifies to

$$p_1(p, 1) = \frac{p_1 p_{1,1}}{p_1 + p_2 \mu_{x(1)-x(0)}}, \quad p_2(p, 1) = \frac{p_1 p_{1,2} + p_2 \mu_{x(1)-x(0)}}{p_1 + p_2 \mu_{x(1)-x(0)}}.$$

Defining

$$T_j^1(p_1, p_2) = \frac{p_1 p_{1,1}}{p_1 + p_2 \mu_j}, \quad T_j^2(p_1, p_2) = \frac{p_1 p_{1,2} + p_2 \mu_j}{p_1 + p_2 \mu_j},$$

we can write the generator A_p ($p = p_1, p_2$) in the form

$$\begin{aligned} A_p g(p) &= \bar{E}^p [g(p(p, 1)) - g(p)] \\ &= p_1 \bar{E}^{e_1} [g(p(p, 1)) - g(p)] + p_2 \bar{E}^{e_2} [g(p(p, 1)) - g(p)] \\ &= \bar{E}^{e_1} [(p_1 + p_2 z_{1,2}(0, 1))(g(p(p, 1)) - g(p))] \\ &= \sum_{j \in N} p_j^{\lambda_1} (p_1 + p_2 \mu_j) [g(T_j^1(p_1, p_2), T_j^2(p_1, p_2)) - g(p_1, p_2)] \quad (4.5) \end{aligned}$$

where we have used the facts

$$\bar{E}^{e_1} [g] = \bar{E}^{e_2} [g z_{1,2}(0, 1)] \quad (g = g(p(p, 1))), \quad z_{1,2}(0, 1) = \mu_{x(1)-x(0)}$$

and the notation $e_1 = (0, 1)$, $e_2 = (1, 0)$.

Define

$$y = p_1/p_2, \quad g(y) = g(p_1, p_2).$$

Then, as easily verified,

$$A_p g(p_1, p_2) = L_y g(y) \quad (4.6)$$

where

$$L_y g(y) = \frac{1}{1+y} \sum_{j \in N} p_j^{\lambda_1} (1+y \mu_j) [g(T_j(y)) - g(y)] \quad (4.7)$$

and

$$T_j(y) = \frac{p_{1,2}}{p_{1,1}} + \frac{\mu_j}{p_{1,1}} y \quad (j \in N). \quad (4.8)$$

We shall impose the monotonicity condition

$$T_j(y) > y \quad (j \in N), \quad (4.9)$$

that is

$$\mu_j > p_{1,1} \quad (j \in N). \quad (4.10)$$

This condition implies that

$$\frac{p_1(p, t)}{p_2(p, t)} > \frac{p_1(p, s)}{p_2(p, s)} \quad \text{if } t > s.$$

It is easily verified that

$$A_p p_2 = -p_{1,2} p_2. \quad (4.11)$$

We shall need Dynkin's formula

$$\bar{E}^{e_1}[g(p(\tau))] - g(p) = \bar{E}^{e_1}\left[\int_0^{\tau-1} A_p g(p(s)) ds\right] \quad (p(0) = e_1) \quad (4.12)$$

where g is any function defined on the discrete set R^* and τ is any bounded $\mathfrak{F}_t$ stopping time.

Let b be any positive number and let τ_b be the first time such that $(p_1(t)/p_2(t)) \geq b$. (Notice $p_1(\tau_b)/p_2(\tau_b)$ is not necessarily equal to b .) Applying (4.12) with $g(p_1, p_2) = p_2$ and $\tau = \tau_b \wedge m$ ($m > 0$) and using (4.11), we conclude that

$$\bar{E}^{e_1}[\tau_b \wedge m] \leq C$$

where C is a constant independent of M . Taking $m \rightarrow \infty$ we conclude that

$$\bar{E}^{e_1}[\tau_b] < \infty. \quad (4.13)$$

Notice that the proof of (4.13) does not exploit the monotonicity assumption (4.9).

We now assume for simplicity that $c_1 = 0$ (but $c_2 > 0$), and set $c = c_2$, $K = K_1$.

Recalling (4.6), (4.7), the q.v.i. (3.6)–(3.8) can be written in terms of the function $V(y) = V(p_1, p_2)$ ($y = p_1/p_2$) in the form

$$L_y V(y) + \frac{cy}{1+y} \geq 0 \quad \text{in } \hat{R}, \quad (4.14)$$

$$V(y) \leq K + \frac{V(0)}{1+y} \quad \text{in } \hat{R}, \quad (4.15)$$

$$\left[L_y V(y) + \frac{cy}{1+y} \right] \left[K + \frac{V(0)}{1+y} - V(y) \right] = 0 \quad \text{in } \hat{R} \quad (4.16)$$

where $\hat{R}$ is the (discrete) range of $p_1(t)/p_2(t)$ when $p_1(0) = 0, p_2(0) = 1$.

THEOREM 4.1. *Let (4.1), (4.9) hold. Then there exist a unique $b \in \hat{R}$, $b > 0$ and a unique function $V(y)$ defined on $\hat{R}$ such that*

$$L_y V(y) + \frac{cy}{1+y} = 0 \quad \text{if } y \in \hat{R}, 0 \leq y < b, \quad (4.17)$$

$$L_y V(y) + \frac{cy}{1+y} > 0 \quad \text{if } y \in \hat{R}, y > b, \quad (4.18)$$

$$V(y) = K + \frac{V(0)}{1+y} \quad \text{if } y \in \hat{R}, y \geq b, \quad (4.19)$$

$$V(y) < K + \frac{V(0)}{1+y} \quad \text{if } y \in \hat{R}, 0 \leq y < b. \quad (4.20)$$

Notice that $V(y)$ is then a solution of the q.v.i. (4.14)–(4.16). In view of (4.13), $\tau_b < \infty$ and therefore Theorem 3.1 can be applied to conclude that

$$V(0) = \inf_{\tau \in \mathcal{Q}} J_x^1(\tau) = J_x^1(\bar{\tau}^1). \quad (4.21)$$

The optimal inspection is then to inspect at time τ_b (given that $p(0) = (0, 1)$); let $p(t)$ start again at $p(0) = (0, 1)$ and again inspect at time τ_b , etc.

PROOF OF THEOREM 4.1. Suppose b is such that (4.17), (4.19) hold. By Dynkin's formula we then get (with $E = \bar{E}^{e_1}$, $P = \bar{P}^{e_1}$, $y(t) = p_1(t)/p_2(t)$)

$$E \left[K + \frac{V(0)}{1 + y(\tau_b)} \right] - V(0) = E \left[- \int_0^{\tau_b-1} \frac{cy(s)}{1 + y(s)} ds \right],$$

or

$$V(0)E \left[\frac{y(\tau_b)}{1 + y(\tau_b)} \right] = K + cE \left[\int_0^{\tau_b-1} \frac{y(s)}{1 + y(s)} ds \right]. \quad (4.22)$$

Setting

$$H(b, dz) = P[y(\tau_b) \in z], \quad (4.23)$$

$$L(b, dz) = \sum_{s < \tau_b-1} P[y(s) \in z], \quad (4.24)$$

we then obtain from (4.22) an expression for $V(0)$:

$$V(0) = \frac{K + c \int \frac{z}{1+z} L(b, dz)}{\int \frac{z}{1+z} H(b, dz)} \equiv Q(b). \quad (4.25)$$

Notation. For any $b \in \hat{R}$ we denote by $\hat{b}$ the number in $\hat{R}$ immediately to the right of b .

LEMMA 4.2. For any $b \in \hat{R}$,

$$L(\hat{b}, db) = P(y(\tau_b) = b). \quad (4.26)$$

PROOF. The left-hand side is equal to

$$\begin{aligned} \sum_{s < \tau_b-1} P(y(s) = b) &= \sum_{s < \tau_b-1} P(y(s) = b) \\ &+ \sum_{\tau_b-1 < s < \tau_b-1} P(y(s) = b). \end{aligned} \quad (4.27)$$

The first sum on the right-hand side is equal to zero (by the definition of τ_b). Since $\tau_b - \tau_b < 1$ by the monotonicity assumption, and $\tau_b - \tau_b = 1$ if and only if $y(\tau_b) = b$, the second sum on the right-hand side of (4.27) is equal to

$$\sum_{s=\tau_b} P(y(s) = b) = P(y(\tau_b) = b),$$

and (4.26) follows.

We also have the relation

$$L(\hat{b}, db) = H(b, db); \quad (4.28)$$

however this relation will not be needed.

LEMMA 4.3. *The following formula holds.*

$$\int \frac{z}{1+z} H(b, dz) = p_{1,2} \int \frac{1}{1+z} L(b, dz). \quad (4.29)$$

PROOF. Since $L_y(1) = 0$,

$$L_y\left(\frac{y}{1+y}\right) = -L_y\left(\frac{1}{1+y}\right) = \frac{p_{1,2}}{1+y}$$

by (4.11). Hence, by Dynkin's formula,

$$E\left[\frac{y(\tau_b)}{1+y(\tau_b)}\right] = E\left[\int_0^{\tau_b-1} \frac{p_{1,2}}{1+y(s)} ds\right].$$

Recalling (4.23) and (4.24), (4.29) follows.

Using Lemma 4.3, we can rewrite the expression $Q(b)$ introduced in (4.25) in the form

$$Q(b) = \frac{K + c \int \frac{z}{1+z} L(b, dz)}{p_{1,2} \int \frac{1}{1+z} L(b, dz)}. \quad (4.30)$$

Our plan now is to show that there is a unique b which minimizes $Q(b)$ and then show that the function V defined by (4.17), (4.19) also satisfies (4.18), (4.20). The uniqueness of the minimal b implies the uniqueness assertion of Theorem 4.1.

For any $b \in \hat{R}$, we compute from (4.30)

$$\begin{aligned} Q(\hat{b}) - Q(b) &= \left[c \int \frac{z}{1+z} L(\hat{b}, dz) - c \int \frac{z}{1+z} L(b, dz) \right] \\ &\quad / \left[p_{1,2} \int \frac{1}{1+z} L(\hat{b}, dz) \right] \\ &\quad + \left[K + c \int \frac{z}{1+z} L(b, dz) \right] \frac{1}{p_{1,2}} \left[\int \frac{1}{1+z} L(b, dz) \right. \\ &\quad \left. - \int \frac{1}{1+z} L(\hat{b}, dz) \right] \\ &\quad / \left[\left(\int \frac{1}{1+z} L(\hat{b}, dz) \right) \cdot \left(\int \frac{1}{1+z} L(b, dz) \right) \right]. \quad (4.31) \end{aligned}$$

By the strict monotonicity of the y -process

$$\int_{z < y} h(z) L(\bar{\bar{y}}, dz) = \int_{z < y} h(z) L(\bar{y}, dz) \quad \text{if } y < \bar{y} < \bar{\bar{y}}. \quad (4.32)$$

Hence

$$\int \frac{1}{1+z} L(\hat{b}, dz) - \int \frac{1}{1+z} L(b, dz) = \frac{1}{1+b} L(\hat{b}, db).$$

Since $b \in \hat{R}$ and the y -process is monotone, $P(y(\tau_b) = b) > 0$ so that, by Lemma 4.2, also $L(\hat{b}, db) > 0$. It follows that

$$\begin{aligned} \operatorname{sgn}[Q(\hat{b}) - Q(b)] &= \operatorname{sgn}\left[cb \int \frac{1}{1+z} L(b, dz) - c \int \frac{z}{1+z} L(b, dz) - K\right] \\ &= \operatorname{sgn}\left[c \int \frac{b-z}{1+z} L(b, dz) - K\right]. \end{aligned} \quad (4.33)$$

Let

$$S(b) = \int \frac{b-z}{1+z} L(b, dz). \quad (4.34)$$

Then, using (4.32),

$$\begin{aligned} S(\hat{b}) - S(b) &= \int \frac{\hat{b}-z}{1+z} L(\hat{b}, dz) - \int \frac{b-z}{1+z} L(b, dz) \\ &= \int \frac{\hat{b}-b}{1+z} L(b, dz) + \frac{1}{1+b} L(\hat{b}, db). \end{aligned} \quad (4.35)$$

The right-hand side is larger than $(\hat{b} - b)L(b, d0) = \hat{b} - b$. Hence $S(b)$ is strictly increasing on $\hat{R}$ and $S(b) \rightarrow \infty$ if $b \rightarrow \infty$.

From (4.33), (4.34) we then conclude that there is a unique b in $\hat{R}$ such that

$$\begin{aligned} Q(\hat{y}) - Q(y) &< 0 \quad \text{if } y < b, y \in \hat{R}, \\ Q(\hat{y}) - Q(y) &> 0 \quad \text{if } y > b, y \in \hat{R}. \end{aligned} \quad (4.36)$$

The point b is then the unique minimum of $Q(y)$, $y \in \hat{R}$.

We next define $V(y)$ for $y > b$, $y \in \hat{R}$ by (4.19) and for $0 < y < b$, $y \in \hat{R}$ by (4.17) (using iteration and the strict monotonicity of the y -process). It remains to show that (4.18) and (4.20) hold.

To prove (4.18) we compute

$$L_y\left(K + \frac{V(0)}{1+y}\right) = \frac{cy}{1+y} - \frac{V(0)p_{1,2}}{1+y}.$$

Thus, (4.18) would follow from

$$cb - V(0)p_{1,2} > 0. \quad (4.37)$$

If we prove that

$$L_y\left(V(y) - K - \frac{V(0)}{1+y}\right) < 0 \quad \text{if } y \in \hat{R}, y < b \quad (4.38)$$

then by the maximum principle (which holds, since the y -process is monotone) and the fact that

$$V(y) - K - \frac{V(0)}{1+y} \begin{cases} = 0 & \text{if } y = b, \\ < 0 & \text{if } y = 0, \end{cases}$$

it follows that (4.20) holds. Since the left-hand side of (4.38) is equal to $cy - V(0)p_{1,2}$, (4.20) is thus a consequence of

$$cy - p_{1,2}V(0) < 0 \quad \text{if } y \in \hat{R}, y < b. \quad (4.39)$$

Thus, to complete the proof of the theorem it remains to prove (4.37), (4.39). Inserting $V(0)$ from (4.25) (or rather (4.30)) into (4.37), (4.39) we find that these two inequalities reduce to

$$cb \int \frac{1}{1+z} L(b, dz) - \left(K + c \int \frac{z}{1+z} L(b, dz) \right) > 0,$$

$$cy \int \frac{1}{1+z} L(b, dz) - \left(K + c \int \frac{z}{1+z} L(b, dz) \right) < 0 \quad (y < b);$$

but these inequalities clearly follow from (4.33) and (4.36).

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